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Nonstationary two-stage multisplitting methods with overlapping blocks

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Abstract

Parallel synchronous two-stage multisplitting methods with overlap for the solution of linear systems of equations are studied. It is shown that under certain hypotheses, the method with overlap is faster, in some measure, than that without overlap. Our results extend the comparison results of multisplittings with overlapping blocks with those of nonoverlapping blocks from (A. Frommer, B. Pohl, A comparison result for multisplittings and wave form relaxation methods, Numer. Linear Algebra Appl. 2 (1995) 335–346) and (M.T. Jones, D.B. Szyld, Two-stage multisplitting methods with overlapping blocks, Numer. Linear Algebra Appl. 3 (1996) 113–124) to the two-stage nonstationary case. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Consider the iterative solution of a large linear system of equations

$$Ax = b, \tag{1.1}$$

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on parallel computers, where $A \in \mathscr{R}^{n,n}$ is nonsingular. O'Leary and White [1] introduced the multisplitting algorithms; Bru et al. [2] proposed parallel chaotic (synchronous and asynchronous) multisplitting methods. Furthermore, Szyld and Jones [3] considered the relationship between two-stage iterative methods (cf. [4–8]) and multisplitting methods. As a result, they introduced two-stage multisplitting methods. Recently, parallel, synchronous and asynchronous two-stage multisplitting methods have been also presented and widely studied by many authors, see, e.g. [9]. Frommer and Pohl [10] first gave some comparision results for multisplitting methods with and without overlap; they showed that certain multisplitting methods based on overlapping blocks yield faster convergence than corresponding ones based on nonoverlapping blocks. Jones and Szyld [11] obtained comparison results between certain stationary two-stage multisplitting methods with and without overlapping blocks.

In this paper we will show similar results to that in [11] for nonstationary two-stage multisplitting methods.

Assume we are given a collection $(M_k, B_k, C_k, N_k, E_k), k = 1, \dots, K$ (which is called a two-stage multisplitting [3]), where $A = M_k - N_k, M_k = B_k - C_k, M_k$ and B_k are nonsingular, E_k are nonnegative diagonal, and $\sum_{k=1}^{K} E_k = I$.

Algorithm 1.1 (*Nonstationary two-stage multisplitting*).

Given the initial vector x^0
For $i = 1, 2, \dots$
  For $k = 1, \dots, K$
    $y_k^0 = x^{i-1}$
    For $j = 0, 1, \dots, p_{k,i} - 1$
      $B_k y_k^{j+1} = C_k y_k^j + N_k x^{i-1} + b$
  $x^i = \sum_{k=1}^{K} E_k y_k^{p_{k,i}}$,
where $p_{k,i}$ are positive integers, which may depend on k and i.

Algorithm 1.1 can be rewritten in the following form [9]

$$x^i = \sum_{k=1}^{K} E_k \left[(B_k^{-1} C_k)^{p_{k,i}} x^{i-1} + \sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j B_k^{-1} (N_k x^{i-1} + b) \right]. \tag{1.2}$$

Let x^* be the solution of Eq. (1.1) and let $e^i = x^* - x^i$ be the error at the ith iteration of Algorithm 1.1. Then

$$e^i = H_i e^{i-1} = H_i H_{i-1} \cdots H_1 e^0, \tag{1.3}$$

where

$$H_i = \sum_{k=1}^{K} E_k \left[(B_k^{-1} C_k)^{p_{k,i}} + \sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j B_k^{-1} N_k \right]. \tag{1.4}$$

We call the matrix $T(i)$

$$T(i) = H_i H_{i-1} \cdots H_1 \tag{1.5}$$

the ith iteration matrix of the nonstationary two-stage multisplitting algorithm resulting from the two-stage multisplitting $(M_k, B_k, C_k, N_k, E_k)$ of A and the positive integer sequence $\{p_{k,i}\}, k = 1, \dots, K, i = 1, 2, \dots$. Henceforth, this nonstationary two-stage multisplitting algorithm is denoted by $\mathrm{NTSM}(M_k, B_k, C_k, N_k, E_k; p_{k,i}), k = 1, \dots, K, i = 1, 2, \dots$.

2. Preliminaries

We begin with some basic notation (cf. [12]).

A matrix $A = (a_{ij}) \in \mathscr{R}^{n,n}$ is called a Z-matrix if $a_{ij} \leqslant 0$ for $i \neq j$. If A is a nonsingular Z-matrix and $A^{-1} \geqslant 0$, then A is called an M-matrix.

A splitting $A = M - N$ of A is called regular if $M^{-1} \geqslant 0$ and $N \geqslant 0$; (left) weak regular if $M^{-1} \geqslant 0$ and $M^{-1}N \geqslant 0$; two-side weak regular if it is (left) weak regular and $NM^{-1} \geqslant 0$; convergent if $\rho(M^{-1}N) < 1$. Here $\rho(C)$ denotes the spectral radius of a matrix C.

Lemma 2.1 [8]. *Given a nonsingular matrix $A \in \mathscr{R}^{n,n}$ and $H \in \mathscr{R}^{n,n}$ such that $I - H$ is nonsingular, there exists a unique pair of matrices F, G, such that $H = F^{-1}G$ and $A = F - G$, where F is nonsingular.*

In the context of this lemma, we say that H induces the splitting $A = F - G$.

A certain monotonic vector norm is defined as follows [13,14]: Let x be a positive vector ($x > 0$). Then

$$\|y\|_x = \inf\{\alpha > 0 \colon -\alpha x \leqslant y \leqslant \alpha x\}, \quad y \in \mathscr{R}^n. \tag{2.1}$$

For a matrix $B \in \mathscr{R}^{n,n}$, $\|B\|_x$ will denote the matrix norm of B induced by the monotonic vector norm $\|\cdot\|_x$. It is easy to show that [14]

$$\|B\|_x = \||B|\|_x = \||B|x\|_x. \tag{2.2}$$

Lemma 2.2. [13]. *Suppose that $A = M_1 - N_1$ and $A = M_2 - N_2$ are two (left) weak regular splittings of monotone matrix A (i.e. $A^{-1} \geqslant 0$), such that $M_2^{-1} \geqslant M_1^{-1}$. If there exists a positive vector x such that $Ax \geqslant 0$, then*

$$\|M_2^{-1}N_2\|_x \leqslant \|M_1^{-1}N_1\|_x. \tag{2.3}$$

In particular, if such a positive vector x is a Perron vector of $M_1^{-1}N_1$, then

$$\rho(M_2^{-1}N_2) \leqslant \rho(M_1^{-1}N_1). \tag{2.4}$$

Definition 2.1 [10]. Let $S_1,\dots,S_K$ be a partition of $\{1,\dots,n\}$, i.e., the sets S_k are pairwise disjoint nonempty subsets of $\{1,\dots,n\}$, so that $\bigcup_{k=1}^K S_k=\{1,\dots,n\}$. Moreover, let $S_k\subseteq T_k\subseteq\{1,\dots,n\}, k=1,\dots,K$, with at least one $k\in\{1,\dots,K\}$, for which $S_k\neq T_k$.

(a) A multisplitting $(M_k,N_k,E_k), k=1,\dots,K$ of $A\in\mathscr{R}^{n,n}$, where

$$\begin{aligned}
&M_k=((M_k)_{ij}) \text{ with } (M_k)_{ij}=\begin{cases} a_{ij} & \text{if } i\in S_k \text{ and } j\in S_k,\\ a_{ii} & \text{if } i=j,\\ 0 & \text{otherwise,}\end{cases}\\
&N_k=M_k-A,\\
&E_k=((E_k)_{ij}) \text{ with } (E_k)_{ij}=\begin{cases} 1 & \text{if } i=j\in S_k,\\ 0 & \text{otherwise}\end{cases}
\end{aligned}\tag{2.5}$$

is called a nonoverlapping block-Jacobi multisplitting of A.

(b) A multisplitting $(\tilde{M}_k,\tilde{N}_k,\tilde{E}_k), k=1,\dots,K$ of $A\in\mathscr{R}^{n,n}$, where

$$\begin{aligned}
&\tilde{M}_k=\left((\tilde{M}_k)_{ij}\right) \text{ with } (\tilde{M}_k)_{ij}=\begin{cases} a_{ij} & \text{if } i\in T_k \text{ and } j\in T_k,\\ a_{ii} & \text{if } i=j,\\ 0 & \text{otherwise,}\end{cases}\\
&\tilde{N}_k=\tilde{M}_k-A,\\
&\tilde{E}_k=\left((\tilde{E}_k)_{ij}\right) \text{ with } (\tilde{E}_k)_{ii}=0 \text{ if } i\notin T_k
\end{aligned}\tag{2.6}$$

is called an overlapping block-Jacobi multisplitting of A.

We remark that due to the structure of the matrices in Eq. (2.5), there exists a permutation matrix P such that

$$\hat{A}\equiv P^{\mathrm{T}}AP=\begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} & \dots & \hat{A}_{1K}\\ \hat{A}_{21} & \hat{A}_{22} & \dots & \hat{A}_{2K}\\ \vdots & \vdots & & \vdots\\ \hat{A}_{K1} & \hat{A}_{K2} & \dots & \hat{A}_{KK}\end{pmatrix},$$

$$\hat{E}_k\equiv P^{\mathrm{T}}E_kP=\begin{pmatrix} 0 &&&&&& \\ & \ddots &&&&& \\ && 0 &&&& \\ &&& I_k &&& \\ &&&& 0 && \\ &&&&& \ddots & \\ &&&&&& 0\end{pmatrix},\tag{2.7}$$

$$\hat{M}_k \equiv P^{\mathrm{T}} M_k P = \begin{pmatrix} \hat{D}_{11} & & & & & & \\ & \ddots & & & & & \\ & & \hat{D}_{k-1,k-1} & & & & \\ & & & \hat{A}_{kk} & & & \\ & & & & \hat{D}_{k+1,k+1} & & \\ & & & & & \ddots & \hat{D}_{KK} \end{pmatrix}, \tag{2.8}$$

$$\hat{N}_k \equiv P^{\mathrm{T}} N_k P = \begin{pmatrix} \hat{D}_{11} - \hat{A}_{11} & \dots & -\hat{A}_{1k} & \dots & -\hat{A}_{1K} \\ \vdots & \ddots & \vdots & & \vdots \\ -\hat{A}_{k1} & \dots & 0 & \dots & -\hat{A}_{kK} \\ \vdots & & \vdots & \ddots & \vdots \\ -\hat{A}_{K1} & \dots & -\hat{A}_{Kk} & \dots & \hat{D}_{KK} - \hat{A}_{KK} \end{pmatrix},$$

where $\hat{A}_{ij} \in \mathscr{R}^{n_i,n_j}, \hat{D}_{jj} = \operatorname{diag}(\hat{A}_{jj})$, while $I_k \in \mathscr{R}^{n_k,n_k}$ is an identity matrix and n_k is the number of entries in S_k. Clearly, $\sum_{j=1}^{K} n_j = n$.

3. Main results

Our aim is to compare convergence rates of nonstationary two-stage multisplitting algorithms. We give the following definition.

Definition 3.1. Let $\{T(i)\}$ and $\{\tilde{T}(i)\}$ (cf. Eqs. (1.4) and (1.5)) be the iteration matrix sequences for $\mathrm{NTSM}(M_k, B_k, C_k, N_k, E_k; p_{k,i})$ and $\mathrm{NTSM}(\tilde{M}_k, \tilde{B}_k, \tilde{C}_k, \tilde{N}_k, \tilde{E}_k; p_{k,i}), k = 1, \dots, K,\ i = 1, 2, \dots,$ respectively. If there exists a positive vector x, such that

$$\|\tilde{T}(i)\|_x \leqslant \|T(i)\|_x, i = 1, 2, \dots,$$

then we say that $\mathrm{NTSM}(\tilde{M}_k, \tilde{B}_k, \tilde{C}_k, \tilde{N}_k, \tilde{E}_k; p_{k,i})$ has faster convergence than $\mathrm{NTSM}(M_k, B_k, C_k, N_k, E_k; p_{k,i}), k = 1, \dots, K, i = 1, 2, \dots$.

Theorem 3.1. *Let $A^{-1} \geqslant 0$. Let (M_k, N_k, E_k) and $(\tilde{M}_k, \tilde{N}_k, E_k)$ denote the nonoverlapping block-Jacobi multisplitting and the overlapping block-Jacobi multisplitting of A, respectively, as defined in Definition 2.1 with the same nonoverlapping weighting matrices E_k. Let the splittings $A = M_k - N_k$ and $A = \tilde{M}_k - \tilde{N}_k$ be regular. Let the splittings $M_k = B_k - C_k$ and $\tilde{M}_k = \tilde{B}_k - \tilde{C}_k$ be two-side weak regular, and defined in such a way that if $(M_k)_{ij} = 0$, then $(B_k)_{ij} = 0$. In addition, let $\tilde{B}_k^{-1} \geqslant B_k^{-1} \geqslant 0$, and $\tilde{B}_k^{-1}\tilde{C}_k \geqslant B_k^{-1}C_k \geqslant 0$. Let $T(i)$ and*

$\tilde{T}(i)$ be the ith iteration matrices of $NTSM(M_k, B_k, C_k, N_k, E_k; p_{k,i})$ and $NTSM(\tilde{M}_k, \tilde{B}_k, \tilde{C}_k, \tilde{N}_k, E_k; p_{k,i})$, respectively. Then for any choice of a positive vector $x > 0$ such that $Ax \geqslant 0$,

$$\|\tilde{T}(i)\|_x \leqslant \|T(i)\|_x, i = 1, 2, \ldots. \tag{3.1}$$

In particular, if, in addition, x (may depend on i) is a Perron vector of $T(i)$, then

$$\rho(\tilde{T}(i)) \leqslant \rho(T(i)). \tag{3.2}$$

Proof. We first note that (cf. Theorem 3.2 in [9]) $\mathrm{NTSM}(M_k, B_k, C_k, N_k, E_k; p_{k,i})$ and $\mathrm{NTSM}(\tilde{M}_k, \tilde{B}_k, \tilde{C}_k, \tilde{N}_k, E_k; p_{k,i})$ are convergent for any initial vector; moreover, for any positive integer i we have

$$H_i v \equiv \left(\sum_{k=1}^{K} E_k \left[(B_k^{-1}C_k)^{p_{k,i}} + \sum_{j=0}^{p_{k,i}-1}(B_k^{-1}C_k)^j B_k^{-1} N_k\right]\right) v \leqslant \theta v, \tag{3.3}$$

$$\tilde{H}_i v \equiv \left(\sum_{k=1}^{K} E_k \left[(\tilde{B}_k^{-1}\tilde{C}_k)^{p_{k,i}} + \sum_{j=0}^{p_{k,i}-1}(\tilde{B}_k^{-1}\tilde{C}_k)^j \tilde{B}_k^{-1} \tilde{N}_k\right]\right) v \leqslant \tilde{\theta} v,$$

where $v = A^{-1}e, e = [1, \ldots, 1]^{\mathrm{T}} \in \mathscr{R}^n, 0 \leqslant \theta, \tilde{\theta} < 1$.

From Eq. (3.3) and the property of the monotonic norm $\|\cdot\|_v$ we have

$$\rho(H_i) \leqslant \|H_i\|_v \leqslant \theta < 1, \rho(\tilde{H}_i) \leqslant \|\tilde{H}_i\|_v \leqslant \tilde{\theta} < 1. \tag{3.4}$$

Thus, we obtain

$$\rho(T(i)) \leqslant \|T(i)\|_v = \|H_i \cdots H_1\|_v \leqslant \prod_{j=1}^{i} \|H_j\|_v < 1,$$

$$\rho(\tilde{T}(i)) \leqslant \|\tilde{T}(i)\|_v = \|\tilde{H}_i \cdots \tilde{H}_1\|_v \leqslant \prod_{j=1}^{i} \|\tilde{H}_j\|_v < 1. \tag{3.5}$$

The assumptions that if $(M_k)_{ij} = 0$, then $(B_k)_{ij} = 0$ imply (cf. Eqs. (2.7) and (2.8))

$$\hat{B}_k \equiv P^{\mathrm{T}} B_k P = \begin{pmatrix} D'_{11} & & & & & & \\ & \ddots & & & & & \\ & & D'_{k-1,k-1} & & & & \\ & & & \hat{B}_{kk} & & & \\ & & & & D'_{k+1,k+1} & & \\ & & & & & \ddots & \\ & & & & & & D'_{KK} \end{pmatrix}, \tag{3.6}$$

$$\hat{C}_k \equiv P^{\mathrm{T}} C_k P = \begin{pmatrix} D''_{11} & & & & & & \\ & \ddots & & & & & \\ & & D''_{k-1,k-1} & & & & \\ & & & \hat{C}_{kk} & & & \\ & & & & D''_{k+1,k+1} & & \\ & & & & & \ddots & \\ & & & & & & D''_{KK} \end{pmatrix}, \tag{3.7}$$

where D'_{jj} and D''_{jj} are diagonal and

$$\hat{M}_{kk} \equiv \hat{A}_{kk} = \hat{B}_{kk} - \hat{C}_{kk}, \hat{D}_{kk} \equiv \ \mathrm{diag}(\hat{A}_{kk}) = D'_{kk} - D''_{kk}. \tag{3.8}$$

From Eq. (3.4) and Lemma 2.1 H_i can induce a unique splitting of $A{:}\, A = F_i - G_i$, such that $H_i = F_i^{-1} G_i = I - F_i^{-1} A$. We now rewrite H_i as follows:

$$\begin{aligned} H_i &= \sum_{k=1}^{K} E_k (B_k^{-1} C_k)^{p_{k,i}} + \sum_{k=1}^{K} E_k \sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j B_k^{-1} (B_k - C_k - A) \\ &= \sum_{k=1}^{K} E_k (B_k^{-1} C_k)^{p_{k,i}} + \sum_{k=1}^{K} E_k \left[\sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j \times (I - B_k^{-1} C_k) - \sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j B_k^{-1} A \right] \\ &= I - \sum_{k=1}^{K} E_k \sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j B_k^{-1} A. \end{aligned}$$

Thus, we have

$$F_i^{-1} = \sum_{k=1}^{K} E_k \sum_{j=0}^{p_{k,i}-1} (B_k^{-1} C_k)^j B_k^{-1} = \sum_{k=1}^{K} E_k \left[I - (B_k^{-1} C_k)^{p_{k,i}} \right] M_k^{-1}. \tag{3.9}$$

Since $F_i^{-1} \geqslant 0$, $H_i \geqslant 0$, the splitting $A = F_i - G_i$ is (left) weak regular. Furthermore, we have

$$\begin{aligned} G_i F_i^{-1} &= (F_i - A) F_i^{-1} = I - \sum_{k=1}^{K} E_k (M_k - N_k) F_i^{-1} \\ &= I - \sum_{k=1}^{K} E_k M_k F_i^{-1} + \sum_{k=1}^{K} E_k N_k F_i^{-1}. \end{aligned} \tag{3.10}$$

For the first two terms in the last line in Eq. (3.10), we have (cf. Eqs. (2.7), (2.8), (3.6), (3.7))

$$P^{\mathrm{T}}\left(I-\sum_{k=1}^{K}E_kM_kF_i^{-1}\right)P$$
$$=I-\sum_{K=1}^{K}\hat{E}_k\hat{M}_k\sum_{j=1}^{K}\hat{E}_j\left[\hat{M}_j^{-1}-(\hat{B}_j^{-1}\hat{C}_j)^{p_{k,i}}\hat{M}_j^{-1}\right]$$
$$=I-\sum_{k=1}^{K}\hat{E}_k\hat{M}_k\sum_{j=1}^{K}\hat{E}_j\hat{M}_j^{-1}+\sum_{k=1}^{K}\hat{E}_k\hat{M}_k\sum_{j=1}^{K}\hat{E}_j(\hat{B}_j^{-1}\hat{C}_j)^{p_{k,i}}\hat{M}_j^{-1}$$
$$=\sum_{k=1}^{K}\hat{E}_k(\hat{C}_k\hat{B}_k^{-1})^{p_{k,i}}\geqslant 0.$$

Thus, we have

$$I-\sum_{k=1}^{K}E_kM_kF_i^{-1}\geqslant 0. \tag{3.11}$$

Combining Eq. (3.11) with Eq. (3.10) we obtain $G_iF_i^{-1}\geqslant 0$. Therefore, the splitting $A=F_i-G_i$ induced by H_i is two-side weak regular.

From Eq. (3.5) and Lemma 2.1, $T(i)$ and $\tilde{T}(i)$ can induce splittings of A.

$$A=F(i)-G(i) \text{ and } A=\tilde{F}(i)-\tilde{G}(i), \tag{3.12}$$

respectively. Obviously,

$$F(i)^{-1}=(I-T(i))A^{-1},\tilde{F}(i)^{-1}=(I-\tilde{T}(i))A^{-1}. \tag{3.13}$$

In case $i=1$, we have

$$F(1)^{-1}\equiv F_1^{-1}\geqslant 0.$$

We now assume $F(i)^{-1}\geqslant 0$. Since

$$T(i+1)=H_{i+1}T(i)=(I-F_{i+1}^{-1}A)(I-F(i)^{-1}A)$$
$$=I-(F_{i+1}^{-1}+F(i)^{-1}-F_{i+1}^{-1}AF(i)^{-1})A,$$

we have

$$F(i+1)^{-1}=F_{i+1}^{-1}+F(i)^{-1}-F_{i+1}^{-1}AF(i)^{-1}$$
$$=F_{i+1}^{-1}+F(i)^{-1}-F_{i+1}^{-1}(F_{i+1}-G_{i+1})F(i)^{-1}$$
$$=F_{i+1}^{-1}+F_{i+1}^{-1}G_{i+1}F(i)^{-1}\geqslant 0.$$

By induction, we have shown that

$$F(i)^{-1}\geqslant 0, i=1,2,\ldots \tag{3.14}$$

Since $G(i)=F(i)-A$ and $F(i)^{-1}=(I-T(i))A^{-1}$ (cf. Eqs. (3.12) and (3.13)), we have

$$
\begin{aligned}
G(i)F(i)^{-1} &= AT(i)A^{-1} = (AH_iA^{-1})\cdots(AH_1A^{-1}) \\
&= G_iF_i^{-1}\cdots G_1F_1^{-1} \geqslant 0.
\end{aligned}
\tag{3.15}
$$

Therefore, splittings $A = F(i) - G(i)$ induced by $T(i), i = 1, 2, \ldots,$ are two-side weak regular.

Finally, we will show that, for each positive integer i,

$$
\tilde{F}(i)^{-1} \geqslant F(i)^{-1}. \tag{3.16}
$$

Firstly, we have (cf. Eq. (3.9))

$$
\begin{aligned}
\tilde{F}_l^{-1} &\equiv \sum_{k=1}^{K} E_k \sum_{j=0}^{p_{k,i}-1} (\tilde{B}_k^{-1}\tilde{C}_k)^j \tilde{B}_k^{-1} \\
&\geqslant \sum_{k=1}^{K} E_k \sum_{j=0}^{p_{k,i}-1} (B_k^{-1}C_k)^j B_k^{-1} \equiv F_l^{-1};
\end{aligned}
\tag{3.17}
$$

thus, Eq. (3.16) holds when $i = 1$.

We now assume that $\tilde{F}(l)^{-1} \geqslant F(l)^{-1}$. Then Eq. (3.13) implies

$$
T(l)A^{-1} \geqslant \tilde{T}(l)A^{-1}. \tag{3.18}
$$

From Eqs. (3.13), (3.15) and (3.18) we have

$$
\begin{aligned}
\tilde{F}(l+1)^{-1} - F(l+1)^{-1} &= \Big(T(l+1) - \tilde{T}(l+1)\Big)A^{-1} \\
&= \Big(H_{l+1}T(l) - \tilde{H}_{l+1}\tilde{T}(l)\Big)A^{-1} \\
&\geqslant \Big(H_{l+1} - \tilde{H}_{l+1}\Big)T(l)A^{-1} \\
&= \Big[(I - \tilde{H}_{l+1}) - (I - H_{l+1})\Big]A^{-1}AT(l)A^{-1} \\
&= \Big(\tilde{F}_{l+1}^{-1} - F_{l+1}^{-1}\Big)G(l)F(l)^{-1} \geqslant 0.
\end{aligned}
\tag{3.19}
$$

Thus, by induction, the proof of Eq. (3.16) is completed.

We have showed that for each positive integer i the splitting $A = F(i) - G(i)$ induced by $T(i)$ is two-side weak regular, while the splitting $A = \tilde{F}(i) - \tilde{G}(i)$ induced by $\tilde{T}(i)$ is (left) weak regular. Furthermore, we have $\tilde{F}(i)^{-1} \geqslant F(i)^{-1}$. Therefore, the assertion of Theorem 3.1 now follows directly from Lemma 2.2. Thus, the proof of the theorem is finished.

When A is an M-matrix, we have the following theorem, the proof of which is almost the same as that of Theorem 3.2 in [11].

Theorem 3.2. *Let A be an M-matrix and D be its diagonal. Let (M_k, N_k, E_k) and $(\tilde{M}_k, \tilde{N}_k, E_k)$ denote the nonoverlapping block-Jacobi multisplitting and the overlapping block-Jacobi multisplitting of A, respectively, as defined in Theorem*

3.1 *with the same nonoverlapping weighting matrices* E_k. *Let* $M_k = B_k - C_k$ *and* $\tilde{M}_k = \tilde{B}_k - \tilde{C}_k$ *be defined in such a way that* $M_k \leqslant B_k \leqslant D$ *and* $\tilde{M}_k \leqslant \tilde{B}_k \leqslant D$. *In addition, let the overlap be such that* $M_k - \tilde{M}_k \geqslant B_k - \tilde{B}_k \geqslant 0$. *Let* $T(i)$ *and* $\tilde{T}(i)$ *be the ith iteration matrices of* $NTSM(M_k, B_k, C_k, N_k, E_k; p_{k,i})$ *and* $NTSM(\tilde{M}_k, \tilde{B}_k, \tilde{C}_k, \tilde{N}_k, E_k; p_{k,i})$, *respectively. Then for any choice of a positive vector* $x > 0$ *such that* $Ax \geqslant 0$,

$$\|\tilde{T}(i)\|_x \leqslant \|T(i)\|_x, i = 1, 2, \ldots$$

In particular, if, in addition, x (may depend on i) is a Perron vector of $T(i)$, *then*

$$\rho(\tilde{T}(i)) \leqslant \rho(T(i)).$$

According to Definition 3.1 we have showed that, under the hypotheses in Theorem 3.1 or Theorem 3.2, $\text{NTSM}(\tilde{M}_k, \tilde{B}_k, \tilde{C}_k, \tilde{N}_k, E_k; p_{k,i})$ has faster convergence than $\text{NTSM}(M_k, B_k, C_k, N_k, E_k; p_{k,i})$.

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